Chapter 3

Vehicle Modeling and Equation Derivation

3.1 Model Overview

As stated earlier, the objective of this research is to develop a general vehicle model that will predict the behavior of a vehicle during a completely defined "combined maneuver". This model is to be useful as a tool for designing vehicle suspensions. The governing equations are derived below with this end in mind.

The vehicle is traveling over a smooth road at a steady state velocity and d'Alembert type forces are applied to the sprung mass center-of-gravity. The d'Alembert's forces are used to represent a given combination of cornering and tractive/ braking effort exerted by the vehicle. This is equivalent to studying the fully dynamic situation at a given instant in time and is referred to as a quasi-static state.

Quasi-static equilibrium requirements on this vehicle can be defined as the constraints necessary to allow cornering during tractive or braking inertial loads. By modeling this situation we must constrain the vehicle such that the appropriate wheel loads occur at all times. This requires that the vehicle will not be constrained in: vertical motion; pitching motion; and rolling motion. These degrees-of-freedoms must must be free to move under the influence of acceleration type forces to distribute the loads and weight transfers appropriately. When the tires are loaded correctly the suspension can assume the correct deflections and wheel orientations.

The remaining degrees-of-freedom: longitudinal motion; lateral motion; and yawing motion, of the vehicle are constrained. The longitudinal constraint corresponds to road load and the steady tractive or braking acceleration; which is a d'Alembert reversed effective force. The lateral constraint, in the nature of a path constraint, also corresponds to a reversed effective lateral inertia force and permits the calculation of turning maneuvers. The only true constraint is that in yaw which enables the vehicle to be held at unbalanced conditions of slip and steering angle. These constraints are applied to the sprung mass to give realistic attitude angles in pitch and roll. The independent variables which are used are front wheel steer angle, DELTA, and vehicle sideslip angle, BETA [16].

3.2 Pragmatic Input Related to Sprung Mass Dynamics

The sprung mass is assumed to be a rigid body with reaction forces acting at the spindle locations. The dynamic equations of motion, with a body-fixed coordinate system, can be found by applying the linear and angular momentum principles, equations (3-1) and (3-2), to the sprung mass seen in Figure 3-1.

$$S\underline{F} = \frac{d\underline{P}}{dt} = m \frac{d\underline{V}}{dt} = m(\dot{\underline{V}} + \underline{w} \times \underline{V}) \qquad (3\text{-}1)$$

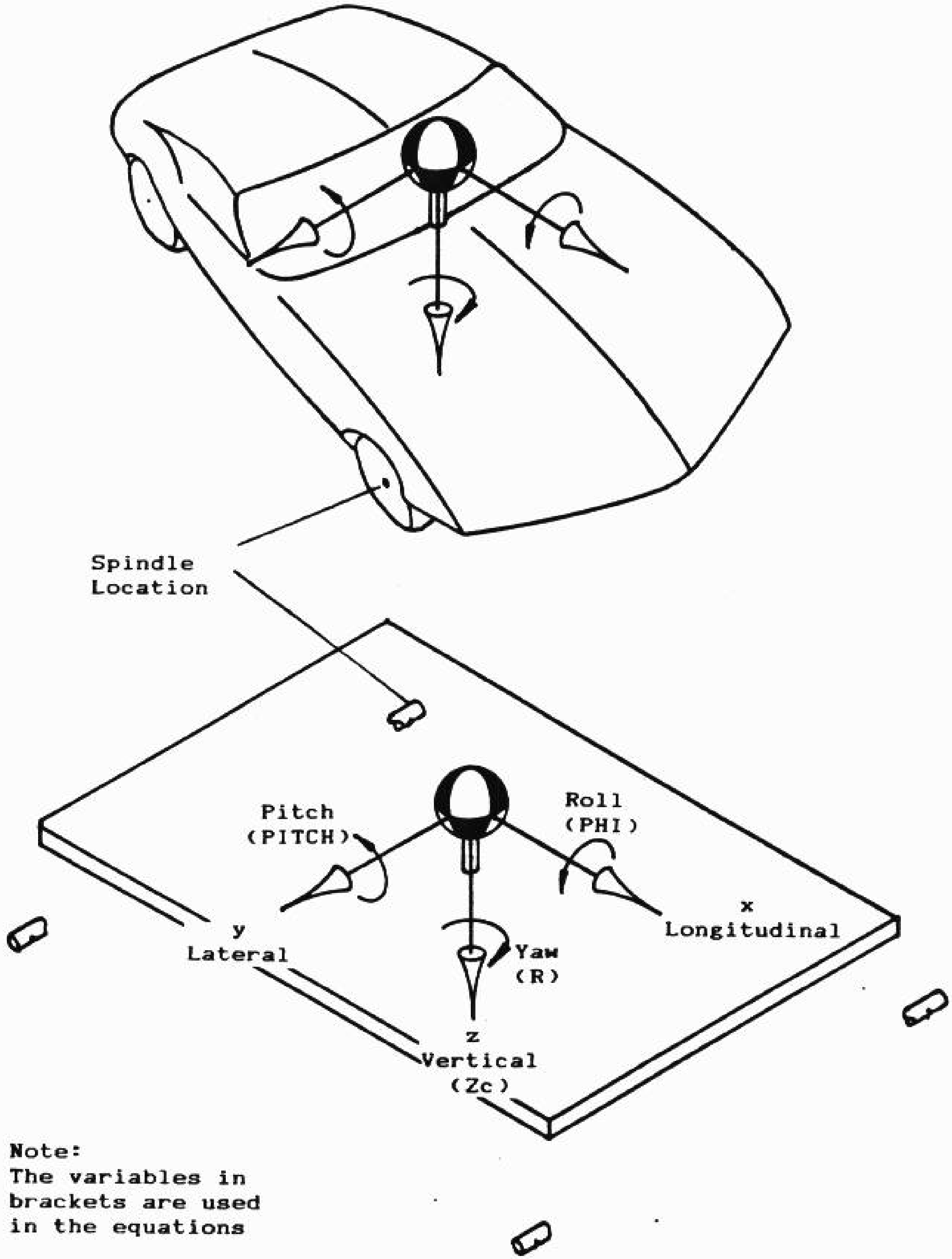


Figure 3-1: Sprung mass body fixed coordinate system

$$\Sigma\underline{T} = \frac{d\underline{H}}{dt} = \underline{I}\,\dot{\underline{w}} + \underline{w} \times \underline{H} \qquad (3\text{-}2)$$

where:

$\Sigma\underline{F}$ = summation-of-generalized-forces vector;
$\dot{\underline{P}}$ = linear momentum vector;
$\dot{\underline{V}}$ and $\underline{V}$ = vehicle linear acceleration and velocity vector;
$\Sigma\underline{T}$ = summation-of-generalized-torques vector;
$\underline{H}$ = angular momentum vector;
$\underline{I}$ = system inertia tensor;
$\dot{\underline{w}}$ and $\underline{w}$ =vehicle angular acceleration and velocity vector.

The cross product terms appear in the above expressions because the coordinate system is fixed to the vehicle. See Crandall [19] for full detail on relative frames of reference.

Under the equilibrium constraints discussed in the previous section the application of the right side of equations (3-1) and (3-2) to the sprung mass becomes very simple. The roll and pitch velocities and accelerations are identically zero for quasi-static equilibrium. Additionally the vertical velocity is zero and the vertical acceleration has only the component due to gravity. These conditions are easily justified if the quasi-static state is equated to a quasi-steady-state condition where all dynamic effects have died out and the vehicle has come to an equilibrium under the externally applied d'Alembert's forces. Then the right side of equations (3-1) and (3-2) for this vehicle become:

$$\Sigma F_x = m\,(\,udot - r\,v\,) \qquad (3\text{-}3)$$

$$\Sigma F_y = m\,(\,vdot + r\,u\,) \qquad (3\text{-}4)$$

$$\Sigma F_z = m\,G \qquad (3\text{-}5)$$

$$\Sigma T_x = I_{xz}\,rdot - I_{yz}\,r^2 \qquad (3\text{-}6)$$

$$\Sigma T_y = I_{yz}\,rdot + I_{xz}\,r^2 \qquad (3\text{-}7)$$

$$\Sigma T_z = I_{zz}\,rdot \qquad (3\text{-}8)$$

where:

SF's = the sum of the reaction forces in the appropriate body fixed direction;
ST's = the sum of the reaction torques about the appropriate body fixed axis;
m = the sprung mass;
u = the velocity in the x (longitudinal) direction;
v = the velocity in the y (lateral) direction;
r = the angular velocity about the z (yaw) direction;
udot = the acceleration in the x direction;
vdot = the acceleration in the y direction;
rdot = the angular acceleration about the y axis;
Izz = the mass moment of inertia about the z axis; and
Ixz and Iyz = the cross products of inertia for yaw.

Recall these values pertain to the vehicle body fixed coordinate system, therefore, inertia properties remain constant.

All the values on the right hand side of equations (3-3) through (3-8) are the externally applied d'Alembert forces. These forces are the input needed to describe the system state. Therefore these constant forces (along with vehicle parameters) completely describe the quasi-static state of the vehicle cornering with tractive or braking effort. However, because vehicle parameters such as cornering stiffness, C.G. location, wheel base, track width, etc. govern the yaw orientation of the vehicle, the orientation of the body fixed input variables (u, v, etc.) to inertial space is unknown. Therefore the orientation of the input forces must be defined as a function of the independent variable BETA, see Figure 3-2a.

A suitable form for this definition shown in Figure 3-2b. The figure shows the vehicle going around a curve of radius RHO. The magnitude of the steady-state velocity, V, is at a right angle to RHO. The angle BETA is the vehicle "sideslip" angle and defines the orientation of the velocity vector to the body fixed x axis. Another vector, A, represents the constant d'Alembert acceleration which is applied to the vehicle. By also defining the angle THETA, the tractive/braking and cornering components of the vehicle acceleration can be separated. This definition of the vehicle state also leads to intuitive understanding. For example, the meaning of the statement, "The car is going around a corner at 100 km/hr and 0.3 G's lateral acceleration" is clear. This is the same as having V =100 km/hr, THETA =90 degrees, and A =0.3 G's. In addition to intuitive appeal this vehicle state definition has other attributes.

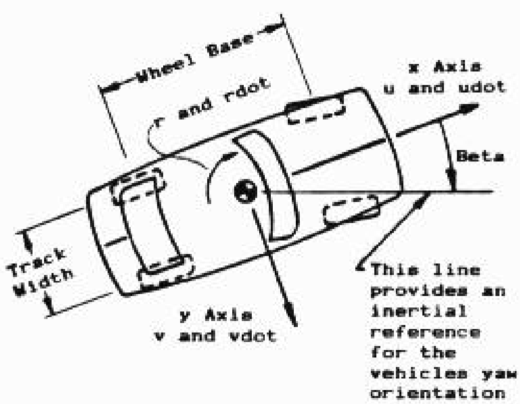


Figure 3-2a: Vehicle yaw orientation to inertial space

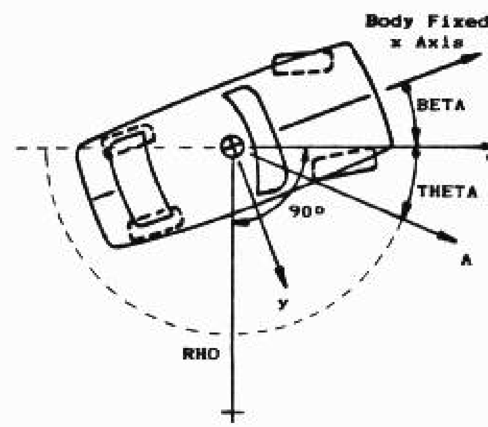


Figure 3-2b: Body fixed variables related to inertial space

By defining THETA relative to BETA rather than the x axis the relation for centripetal acceleration may be utilized to define the length of RHO. From the figure,

$$A_{CENTRIPETAL} = \frac{V^2}{RHO} = A \sin(THETA)$$

therefore,

$$RHO = \frac{V^2}{A \sin(THETA)} \quad (3\text{-}9).$$

Also for quasi-steady-state the yaw velocity, r, and yaw acceleration, rdot are found by assuming,

$$r = \frac{V}{RHO}.$$

By using equation (3-9) and referring to the figure,

$$r = \frac{V}{(V^2/\ A \sin(THETA))} = \frac{A \sin(THETA)}{V}. \quad (3\text{-}10)$$

Next rdot is found by taking the time derivative of (3-10) as follows:

$$rdot = \frac{d}{dt} \frac{A \sin(THETA)}{V} = \frac{-(A \sin(THETA))}{V^2} \frac{dV}{dt}$$

From Figure 3.2b dV/dt equals A cos(THETA) which is the component of the constant acceleration in the same direction as the velocity vector, so:

$$rdot = \frac{-A^2}{V^2} \sin(THETA) \cos(THETA). \quad (3\text{-}11)$$

The remaining input parameters are also defined in terms of BETA as is shown in the figure.

$$u = V \cos(BETA) \quad (3\text{-}12)$$

$$v = V \sin(BETA) \quad (3\text{-}13)$$

$$udot = A \cos(THETA + BETA) \quad (3\text{-}14)$$

$$vdot = A \sin(THETA + BETA) \quad (3\text{-}15)$$

The specification of the system state is now completely defined as a function of one independent variable, BETA. Now any combination of lateral and longitudinal acceleration can be prescribed for a given steady-state-velocity. From Figure 3-2b, the A vector can be rotated from 0 to 180 degrees and will represent any case, from pure tractive acceleration, through pure cornering, to pure braking. Based on the above definition any equilibrium position or sprung mass reaction force can be calculated.

3.3 Force - Moment Reaction Summation on the Sprung Mass

The vehicle model must be general enough to be useful in the early stages of design. In this case the model is a lumped mass representation of an automotive vehicle. The sprung mass has the six rigid body degrees-of-freedom (dof) and each unsprung mass has an essentially vertical dof. The last dof is the steering angle for a total of eleven dof's. However to promote generality, this model uses spindle forces as the unknowns (three at each spindle) and the dof's described above are displacements. The only displacement unknowns used in this model are the steer angle, DELTA; and the vehicle sideslip angle, BETA. The forces were favored as unknowns for the following reasons:

- models of different suspensions are easier to implement, especially when anti-features are important;
- system equations-of-motion are in a simpler form;
- the drive train constraints are easier to describe in terms of forces;and
- the lateral tire forces are functions of longitudinal and normal spindle forced not displacements.

As a result of this selection there are twelve spindle forces plus BETA and DELTA, for a total of fourteen unknowns.

Generality is maintained by defining the locations of spindles with respect to the sprung mass in a pragmatic manner. This is done by defining the vertical (z direction) distance between the spindles and the center-of-gravity; these locations are known very early during vehicle conception. For each spindle this distance has three components:

Z_C = vertical deflection of the center-of-gravity;

$Z_{S\_\_}$ = vertical deflection of a given spindle; and

H = vertical distance between the C.G. and the plane containing all four spindles at design position.

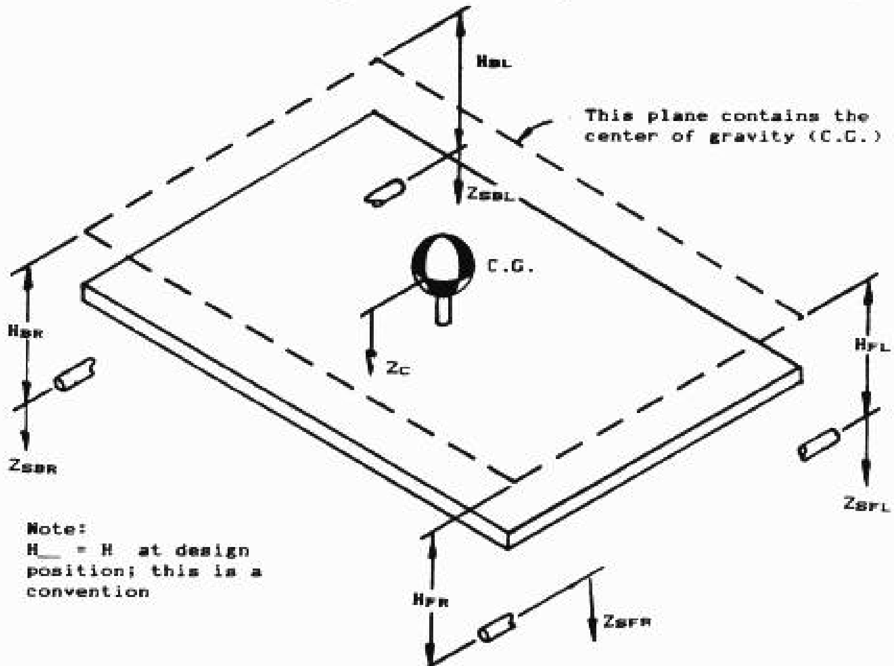


Figure 3-3: Center of gravity to spindle vertical distance

The first two of these parameters, Z_C and $Z_{S\_\_}$, are calculated in the model. The H value is found by subtracting the static-loaded-radius of the tires from the vertical position of the C.G. relative to ground at design position. Consequently, the "C.G. to spindle distance" is defined for each spindle in Figure 3-3.

$$H_{FR} = H - Z_C + Z_{SFR} \qquad (3\text{-}16)$$

$$H_{FL} = H - Z_C + Z_{SFL} \qquad (3\text{-}17)$$

$$H_{BR} = H - Z_C + Z_{SBR} \qquad (3\text{-}18)$$

$$H_{BL} = H - Z_C + Z_{SBL} \qquad (3\text{-}19)$$

The subscripts used throughout this document will be explained as needed except for those used to express a specific corner of the vehicle. In this case a standard convention has been adopted. The subscripts FR, FL, BR, and BL refer to the front-right, front-left, back-right, and back-left respectively. This convention is applied throughout the whole document.

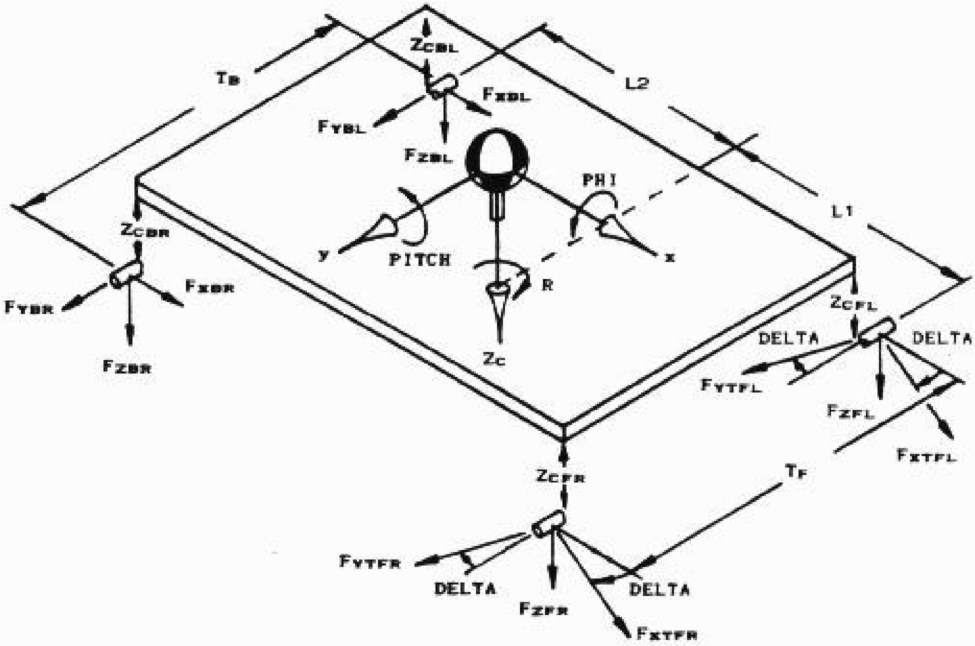


Figure 3-4: Sprung mass free body diagram

Figure 3-4 shows a pseudo free body diagram of the sprung mass. The reaction forces at each spindle have three orthogonal components. At the back, each component is oriented so it is parallel to the respective body-fixed axis. In the front, the X and Y direction force are similarly oriented but rotated an angle DELTA, the steer angle. This is done to facilitate interaction between the main vehicle model and the tire subroutine. Specifically, no trigonometric conversions are needed to send forces to the tire subroutine because the forces aligned with a standard tire coordinate system.

The summation of forces and moments can now be defined by assuming the vehicle is symmetric from left to right; knowing the front and back track widths, T_F and T_B; and the fore/aft location of the C.G., L_1 and L_2. These summations are the left hand side of equation (3-1) and (3-2). Referring to Figure 3-4 and noting the forces are exerted on the body by the suspension we have:

$$SF_X = F_{XBR} + F_{XBL} + F_{XTFR} \cos(DELTA) - F_{YTFR} \sin(DELTA) + F_{XTFL} \cos(DELTA) - F_{YTFL} \sin(DELTA) \quad (3-20)$$

$$SF_Y = F_{YBR} + F_{YBL} + F_{YTFR} \cos(DELTA) + F_{XTFR} \sin(DELTA) + F_{YTFL} \cos(DELTA) + F_{XTFL} \sin(DELTA) \quad (3-21)$$

$$SF_Z = F_{ZBR} + F_{ZBL} + F_{ZFR} + F_{ZFL} \quad (3-22)$$

$$\begin{aligned} ST_X = {} & (F_{ZFR} - F_{ZFL}) T_F/2 + (F_{ZBR} - F_{ZBL}) T_B/2 - F_{YBR} H_{BR} \\ & - (F_{XTFL} \sin(DELTA) + F_{YTFL} \cos(DELTA)) H_{FL} \\ & - (F_{XTFR} \sin(DELTA) + F_{YTFR} \cos(DELTA)) H_{FR} \\ & - F_{YBL} H_{BL} \end{aligned} \quad (3-23)$$

$$\begin{aligned} ST_Y = {} & (F_{ZBR} + F_{ZBL}) L2 - (F_{ZFR} - F_{ZFL}) L1 + F_{XBR} H_{BR} \\ & + (F_{XTFR} \cos(DELTA) - F_{YTFR} \sin(DELTA)) H_{FR} \\ & + (F_{XTFL} \cos(DELTA) - F_{YTFL} \sin(DELTA)) H_{FL} \\ & + F_{XBL} H_{BL} \end{aligned} \quad (3-24)$$

$$\begin{aligned} ST_Z = {} & (F_{XBL} - F_{XBL}) T_B/2 - (F_{YBR} + F_{YBL}) L2 \\ & + (F_{YTFR} + F_{YTFL}) L1 \cos(DELTA) \\ & + (F_{XTFL} + F_{XTFR}) L1 \sin(DELTA) \end{aligned}$$

$+ (F_{XTFL} - F_{XTFR}) \cos(DELTA)\ T_F/2$

$+ (F_{YTFR} - F_{YTFL}) \sin(DELTA)\ T_F/2$ (3-25)

By combining equations (3-20) to (3-25) with (3-3) to (3-8) the equations of motions of the sprung mass are complete.

3.4 Suspension Force and Displacement Relations

The reaction forces on the sprung mass deflect the suspension and tire springs to generate an equilibrium position of the total vehicle. The constitutive relations for the suspension and tire springs must be generated. These constitutive relations will then be used to solve for the twelve unknown forces at the spindles by transforming them into vehicle constraint relations.

First we define the vertical displacement of each suspension spring in terms of the over-all vehicle geometry, track width, wheel base, spindle displacement, and the sprung-mass rigid-body displacements (roll, pitch, and bounce). These relations are shown below in linear form because normal values of roll and pitch rarely exceed ten degrees (refer to Figure 3-4).

$Z_{CBR} = Z_C + (PITCH)\ L2 + PHI\ (T_B/2) - Z_{SBR}$ (3-26)

$Z_{CBL} = Z_C + (PITCH)\ L2 - PHI\ (T_B/2) - Z_{SBL}$ (3-27)

$Z_{CFR} = Z_C - (PITCH)\ L1 + PHI\ (T_F/2) - Z_{SFR}$ (3-28)

$Z_{CFL} = Z_C - (PITCH)\ L1 - PHI\ (T_F/2) - Z_{SFL}$ (3-29)

where $Z_{C\_\_}$ is the relative displacement between the sprung mass and the wheel spindle, and $Z_{S\_\_}$ is the vertical spindle displacement relative to ground.

By defining the displacements at each spindle in this manner an anti-roll bar can easily be incorporated into the model. The roll bar is a spring that works in torsion and adds roll stiffness to the vehicle. A roll bar can be modelled as shown in Figure 3-5. The restoring forces generated by the roll bar are a function of the difference between spring displacements left to right. Therefore the total force generated on each spindle has two terms; one that acts in roll and one that acts in bounce. With the spring displacements defined above, the relation for the vertical forces acting on the spindles by the suspension can be generated.

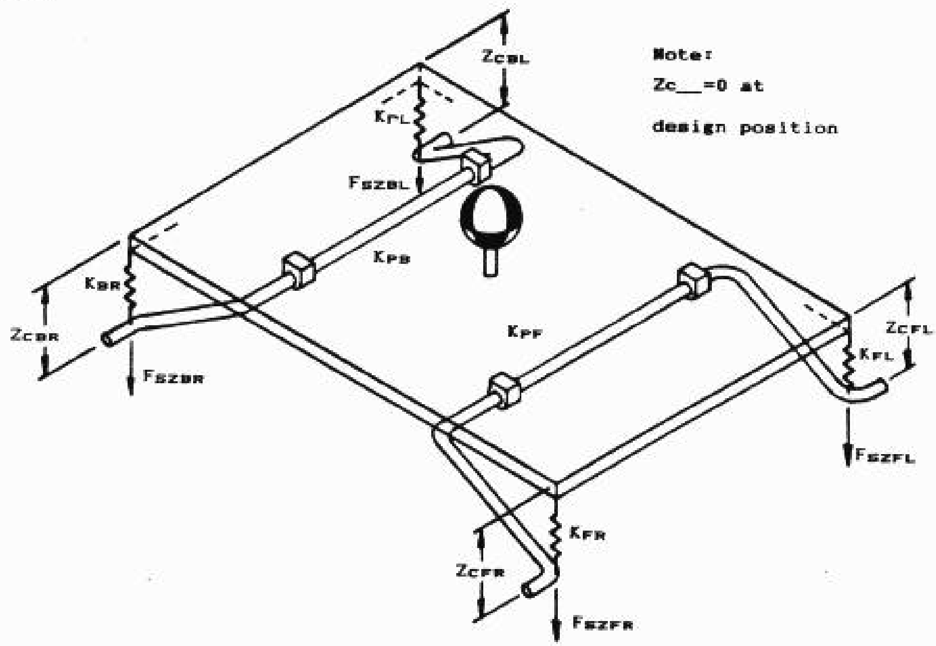


Figure 3-5: Roll bar and suspension spring forces

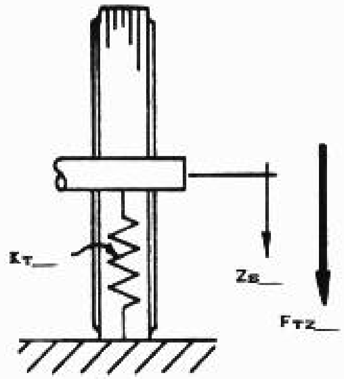
$$F_{SZBR} = K_{BR}\, Z_{CBR} + K_{PB}\,(Z_{CBR} - Z_{CBL}) \qquad (3\text{-}30)$$

$$F_{SZBL} = K_{BL}\, Z_{CBL} + K_{PB}\,(Z_{CBL} - Z_{CBR}) \qquad (3\text{-}31)$$

$$F_{SZFR} = K_{FR}\, Z_{CFR} + K_{PF}\,(Z_{CFR} - Z_{CFL}) \qquad (3\text{-}32)$$

$$F_{SZFL} = K_{FL}\, Z_{CFL} + K_{PF}\,(Z_{CFL} - Z_{CFR}) \qquad (3\text{-}33)$$

The constitutive relations for the radial tire spring can also be written using the same displacement definitions. Referring to Figure 3-6 the vertical forces applied on the spindle by the tire spring are:



$$F_{TZBR} = - K_{TB}\ Z_{SBR} \quad (3\text{-}34)$$

$$F_{TZBL} = - K_{TB}\ Z_{SBL} \quad (3\text{-}35)$$

$$F_{TZFR} = - K_{TF}\ Z_{SFR} \quad (3\text{-}36a)$$

$$F_{TZFL} = - K_{TF}\ Z_{SFL} \quad (3\text{-}36b)$$

Figure 3-6: Radial tire spring

Next, the above constitutive relations will be used to calculate the sprung mass rigid body deflections. To do this we must realize that the suspension spring rates and the radial tire rates are equal from one side to the other. This simplifies the relations and is the case for all normal vehicles. Therefore:

$$K_F = K_{FR} = K_{FL}$$

$$K_B = K_{BR} = K_{BL} \quad (3\text{-}37a)$$

$$K_{TF} = K_{TFR} = K_{TFL}$$

$$K_{TB} = K_{TBR} = K_{TBL} \quad (3\text{-}37b)$$

By adding equation (3-32) to (3-33), noting that from Figure 3-7:

$$F_{Z\_\_} = -F_{SZ\_\_} \quad \text{(where \_\_ = BR,BL,FR,FL)} \quad (3\text{-}38a)$$

$$F_{Z\_\_} = F_{TZ\_\_} \quad \text{(where \_\_ = BR,BL,FR,FL)} \quad (3\text{-}38b)$$

and using equations (3-28) and (3-29) the relation for pitch is:

$$PITCH = \frac{2\ Z_C - Z_{SFL} - Z_{SFR}}{2\quad L1} - \frac{(F_{ZFL} + F_{ZFR})}{2\ K_F\ L1} \quad (3\text{-}39)$$

By adding equation (3-30) to (3-31), and using equations (3-26), (3-27), (3-37), (3-38) and (3-39) vehicle bounce can be similarly calculated.

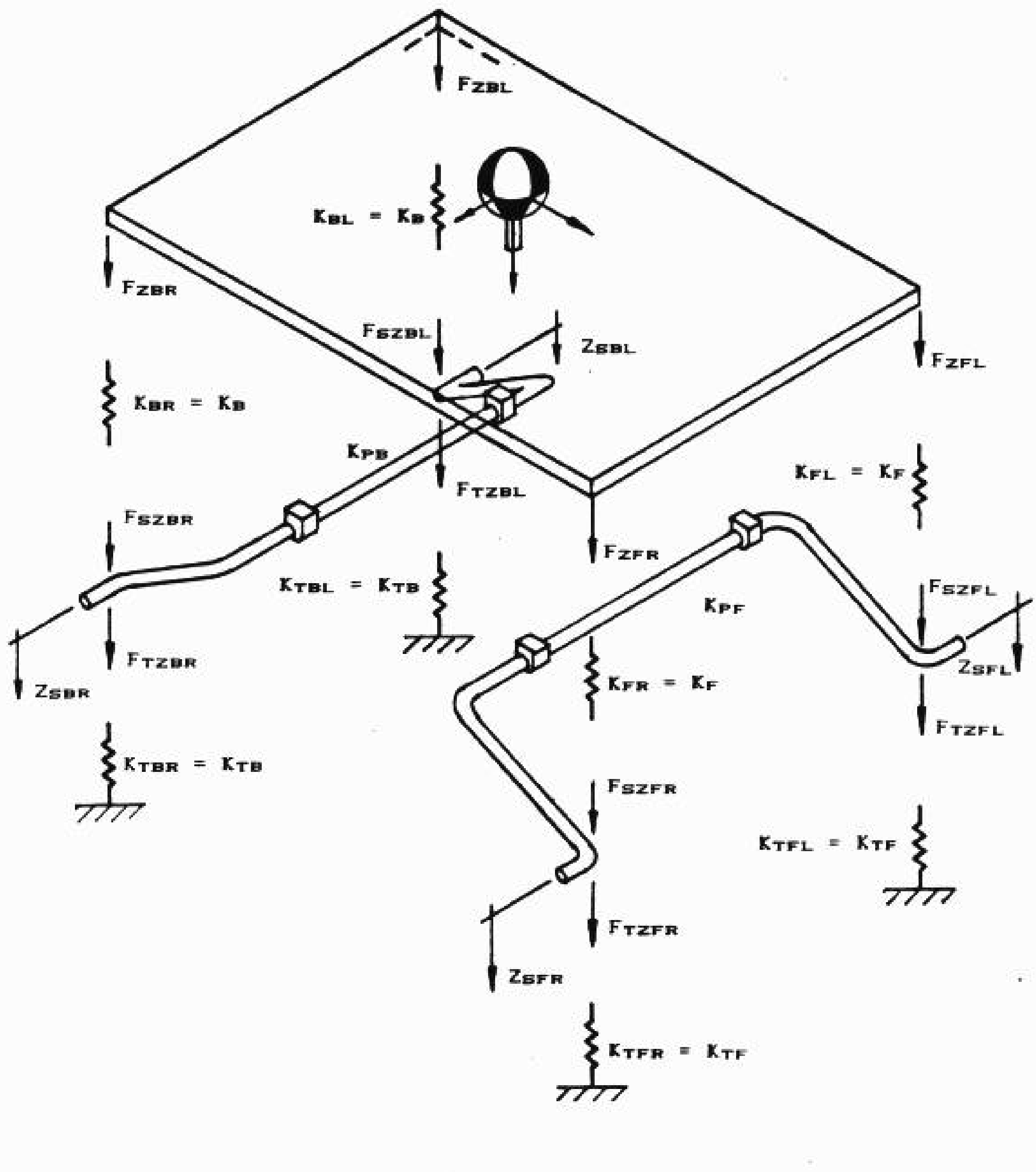


Figure 3-7: Sprung mass F.B.D. to show force interaction

$$Z_C = [\ L2\ (Z_{SFL} + Z_{SFR}) + L1\ (Z_{SBR} + Z_{SBL})$$
$$- L2\ (F_{ZFL} + F_{ZFR})\ /\ K_F$$
$$- L1\ (F_{ZBR} + F_{ZBL})\ /\ K_B\ ]\ /\ (2\ L1 + 2\ L2) \qquad (3\text{-}40)$$

By subtracting equation (3-32) from (3-33) and using equations (3-28), (3-29), (3-37), and (3-38) the equation for roll is:

$$PHI = \frac{(F_{ZFL} - F_{ZFR})\ (K_{TF} - K_F - 2\ K_{PF})}{T_F\ \ K_{TF}\ \ (K_F + K_{PF})} \qquad (3\text{-}41)$$

3.5 Sprung Mass Rigid Body Constraint

Since the spindle forces are used as unknowns in the sprung mass equations-of-motion, the model mathematics do not reflect the fact that the sprung mass is assumed to be a rigid body. Therefore, an additional constraint relation must be generated that enforces the rigid body assumption. If the forces are written in terms of rigid body coordinates, this is not required. This is best illustrated through an example. Figure 3-8 shows a

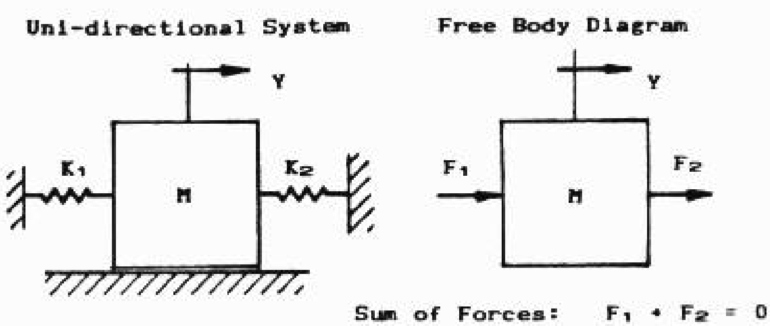


Figure 3-8: One dof system to demonstrate rigid body constraint

simple unidirectional system which consists of a mass supported between two springs. If the forces are summed on the mass, the result is a single equation in terms of two unknown spring forces

F1 and F2. Because there is one equation and two unknowns the constitutive relations of the springs must be used to generate another equation in terms of the unknown forces. These relations are $F1 = K1\ Y$ and $F2 = K2\ Y$. Therefore $F1/K1 = F2/K2$ is the second equation necessary to solve the problem. The second equation represents the rigid body assumption for the mass.

A similar equation can be generated for the vehicle model. The necessary constitutive relations are: the vertical spindle forces exerted on the spindles by the suspension, equations (3-30) through (3-33); and the forces exerted on the spindle by the tires, (3-34) through (3-36b). By combining these equations and also using (3-26) through (3-29) and (3-38) the rigid body constraint for the vehicle is:

$$\frac{(F_{ZFL} - F_{ZFR})}{(F_{ZBL} - F_{ZBR})} = \frac{(K_{TB} - K_B - 2\ K_{PB})\ K_{TF}\ (K_F + 2\ K_{PF})\ T_F}{(K_{TF} - K_F - 2\ K_{PF})\ K_{TB}\ (K_B + 2\ K_{PB})\ T_B} \quad (3\text{-}42)$$

Notice the the right hand side of equation (3-42) is a constant (call it "C") in terms of vehicle parameters. The left hand side is a ratio of the lateral load transfer in the front over the lateral load transfer in the back. By coincidence this equation is very similar to a parameter called TLLTD, Total Lateral Load Transfer Distribution. Very often this TLLTD parameter is used to characterize the understeer/oversteer behavior of a vehicle. In other words, the understeer/oversteer traits of a vehicle can be altered by changing the TLLTD. This similarity can be further investigated through the use of this model.

3.6 Drive Train System Force Constraints

Further constraints on the spindle forces are generated by imposing the tractive/braking, or longitudinal characteristics of the vehicle as described in section 3.1. These characteristics are a function of the type of drive train installed; FWD (front wheel drive), or RWD (rear wheel drive), and closely related are the constraints imposed by the braking system.

By assuming the vehicle is traveling along a smooth road on a relatively large radius curve the drive train differential can be modelled as ideal. This implies that the torque applied to each wheel is equal from side to side. Therefore the longitudinal forces on the spindles of the two drive wheels will be equal also, assuming the loaded radius of the tires is equal.

Under the rear wheel drive condition the longitudinal forces, F_{XTFR} and F_{XTFL}, are equal to zero; neglecting drag, engine braking, etc. The back longitudinal forces are equal to each other, $F_{XBR} = F_{XBL}$. By substituting these conditions into (3-20), the result is the force constraints for a rear wheel drive vehicle.

$$F_{XTFR} = 0.0 \qquad (3\text{-}43a)$$

$$F_{XTFL} = 0.0 \qquad (3\text{-}43b)$$

$$F_{XBL} = F_{XBR} \qquad (3\text{-}43c)$$

$$F_{XBR} = 1/2\,(\;SF_X + (\,F_{YTFR} + F_{YTFL})\,\sin(DELTA)) \qquad (3\text{-}43d)$$

Under the front wheel drive condition the longitudinal forces, F_{XBR} and F_{XBL}, are equal to zero. The front longitudinal forces are equal to each other, $F_{XTFR} = F_{XTFL}$. By substituting these conditions into (3-20), the result is the force constraints for a front wheel drive vehicle.

$$F_{XBL} = 0.0 \qquad (3\text{-}44a)$$

$$F_{XBR} = 0.0 \qquad (3\text{-}44b)$$

$$F_{XTFL} = F_{XTFR} \qquad (3\text{-}44c)$$

$$F_{XTFR} = \frac{(SF_X + (F_{YTFR} + F_{YTFL}) \sin(DELTA))}{2 \cos(DELTA)} \qquad (3\text{-}44d)$$

When the vehicle is braking, longitudinal forces are exerted at all four spindle locations. These forces are directly related to the brake fluid pressure that is delivered to each wheel. Based on the distribution of this pressure the force constraint for braking is derived.

A typical brake system has a front-to-back proportioning valve which biases fluid pressure delivery in favor of the front brakes. In addition, there is no bias from right to left so the force constraint can be generated using only one side of the vehicle. Finally, resulting forces are just equated from right to left; $F_{XTFL} = F_{XTFR}$ and $F_{XBL} = F_{XBR}$. This is done by realizing the total right side brake force is equal to $(F_{XBR} + F_{XTFR})$. Notice, since F_{XTFR} is aligned with the tire coordinate system the brake force delivered to the front wheel is not a function of DELTA (see Figure 3-4). So, if BP is the fraction of the brake force delivered to the front wheel the relation:

$$F_{XTFR} = BP\ (F_{XBR} + F_{XTFR}) \qquad \text{where } (0 < BP < 1.0)$$

is the front brake force, which can be solved for F_{XTFR} to get equation (3-45c). So the force constraints for braking are:

$$F_{XTFL} = F_{XTFR} \qquad (3\text{-}45a)$$

$$F_{XBL} = F_{XBR} \qquad (3\text{-}45b)$$

$$F_{XTFR} = \frac{F_{XBR}\ BP}{(1 - BP)} \qquad (3\text{-}45c)$$

By substituting (3-45c) into (3-20) the final brake constraint relation is:

$$F_{XBR} = \frac{\left(SF_X + (F_{YTFL} + F_{YTFR}) \sin(DELTA) \right)}{\dfrac{2\,(1 + (BP\cos(DELTA)\,)\,)}{(1 - BP)}} \qquad (3\text{-}45d)$$

3.7 Kinematic Relation Between Vehicle Sideslip and Tire Slip Angles.

The lateral force generated by a tire is a function of the slip angle as described in chapter 2. Therefore the slip angles are calculated so they may be used as input to the tire model.

The steering linkage dictates a constraint between the left and right front steer angles. This constraint can be very complex if all the steering geometry is included. A very good approximation can be made as follows; at moderate speeds the steer angles can be assumed to be equal. This is because the slip angle phenomenon has a much greater effect on cornering behavior than the difference between steer angles.

Steer effects at the back wheels as a function of suspension travel are neglected in this model. These effects can be included in the suspension subroutine if desired.

The quasi-steady-state assumption implies that the velocity of any point on the vehicle must be directed at a right angle to the radius of the turn. This Implies the center-of-each-tire-contact-patch velocity must also be at a right angle. This is described in section 3.1 as a path constraint. Therefore, Figure 3-9 shows there is a kinematic relationship between the slip angles (ALF__), steer angle (DELTA), vehicle sideslip (BETA), and the center of the turn. Note that positive rotation is clockwise and the slip angles are shown in a negative position.

The steer angle DELTA defines the location of the tire plane relative to the x (longitudinal) direction. The angle between the tire plane and the contact-patch-velocity heading is the slip angle; either ALF_{FR} or ALF_{FL}. This same velocity heading can be defined as a function of the vehicle sideslip, BETA. This functional relationship only depends on the location of the C.G. with respect to the center of the tire contact patch in the plan view. By defining the velocity of the contact patch in this manner equations can be derived for the slip angles in terms of BETA and DELTA, refer to Figure 3-9a.

$$ALF_{FL} = BETA - DELTA + XI \qquad (3\text{-}46)$$

$$ALF_{FR} = BETA - DELTA + ETA \qquad (3\text{-}47)$$

To find the angles XI and ETA needed in these relationships the law of sines and the law of cosines are used. First the law of cosines is used to find RHO_{FR} and RHO_{FL}.

$$RHO_{FR} = [\, RHO^2 + T_F^2/4 + L1^2 - 2\, RHO\, (T_F^2/4 + L1^2)^{1/2} \cos(PI/2 + BETA - \tan^{-1}(T_F/2L1))]^{1/2} \qquad (3\text{-}48)$$

$$RHO_{FL} = [\, RHO^2 + T_F^2/4 + L1^2 - 2\, RHO\, (T_F^2/4 + L1^2)^{1/2} \cos(PI/2 + BETA + \tan^{-1}(T_F/2L1))]^{1/2} \qquad (3\text{-}49)$$

Now use these two relations and the law of sines to find the XI and ETA.

$$ETA = \sin^{-1}[\, (T_F^2/4 + L1^2)^{1/2} \sin(PI/2 + BETA - \tan^{-1}(T_F/2L1)) / RHO_{FR}\,] \qquad (3\text{-}50)$$

$$XI = \sin^{-1}[\, (T_F^2/4 + L1^2)^{1/2} \sin(PI/2 + BETA + \tan^{-1}(T_F/2L1)) / RHO_{FL}\,] \qquad (3\text{-}51)$$

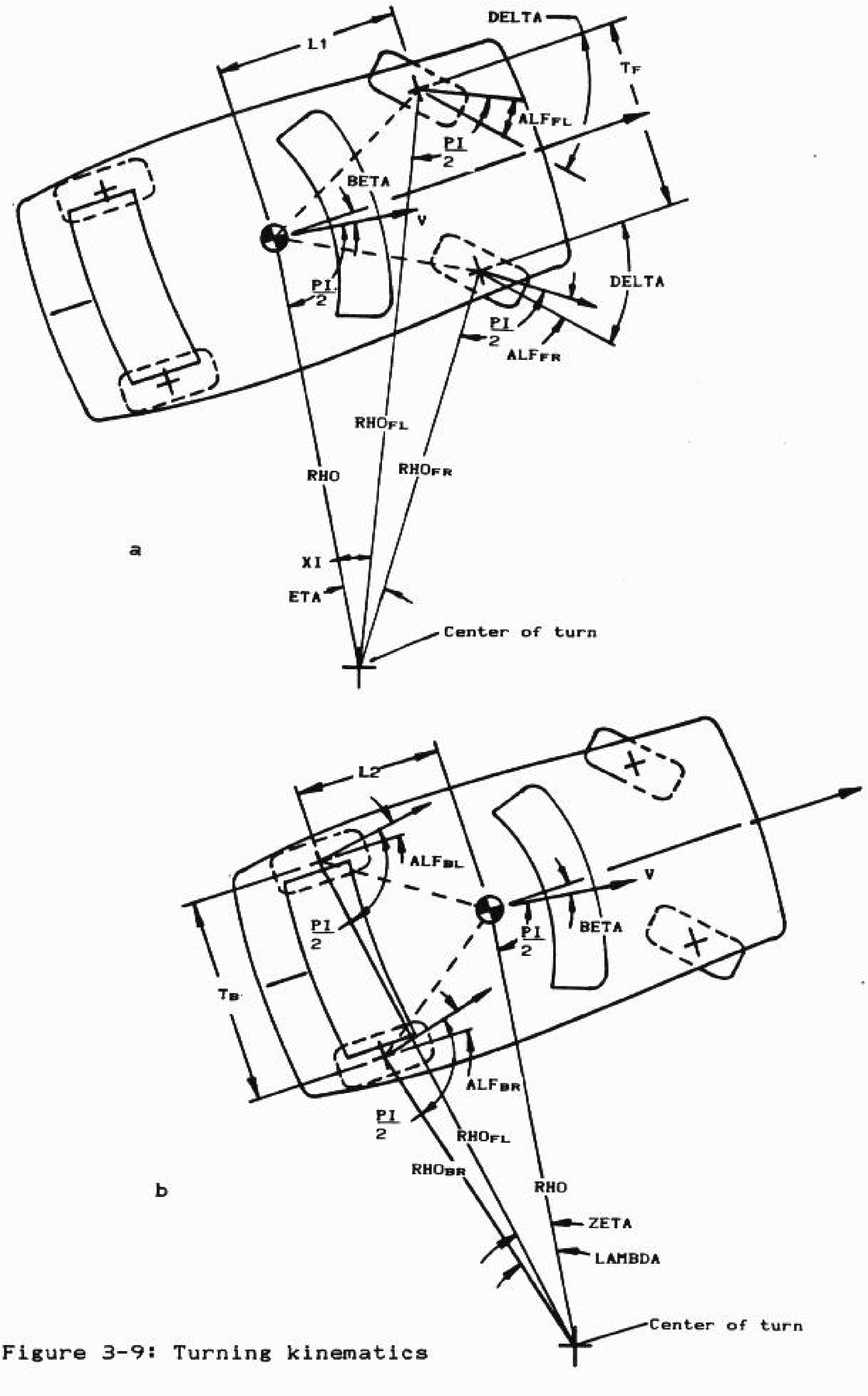


Figure 3-9: Turning kinematics

The equations for the back slip angles ALF_{BL} and ALF_{BR} can be found similarly by referring to Figure 3-9b. Since we have neglected the steer effects at the back axle, ALF_{BL} and ALF_{BR} are not functions of DELTA.

$$ALF_{BL} = BETA - ZETA \tag{3-52}$$

$$ALF_{BR} = BETA - LAMBDA \tag{3-53}$$

$$RHO_{BR} = [\, RHO^2 + T_B^2/4 + L2^2 - 2\, RHO\, (T_B^2/4 + L2^2)^{1/2} \cos(PI/2 - BETA - \tan^{-1}(T_B/2L2))]^{1/2} \tag{3-54}$$

$$RHO_{FL} = [\, RHO^2 + T_B^2/4 + L2^2 - 2\, RHO\, (T_B^2/4 + L2^2)^{1/2} \cos(PI/2 - BETA + \tan^{-1}(T_B/2L2))]^{1/2} \tag{3-55}$$

$$ZETA = \sin^{-1}[\, (T_B^2/4 + L2^2)^{1/2} \sin(PI/2 - BETA - \tan^{-1}(T_B/2L2))/\, RHO_{BR}\,] \tag{3-56}$$

$$LAMBDA = \sin^{-1}[\, (T_B^2/4 + L2^2)^{1/2} \sin(PI/2 - BETA + \tan^{-1}(T_B/2L2))/\, RHO_{BL}\,] \tag{3-57}$$

3.8 Tire Model

By generating the vehicle model in a general nature many tire models exist that can be implemented. Most analytic tire models have a standardized input/output format. In general, the tire model input consists of a slip angle, a normal load, and a camber angle. Based on these values (and a characterization of the specific tire being used) the output will contain the lateral force, aligning torque, and overturning moment that the tire can generate.

The complexity of an analytical tire model may vary anywhere from simple linear coefficients to very complex non-linear relations. In the development of the computer model a simple linear tire model is utilized. The linear tire model equation is:

$$F_Y = CALPHA\ ALF \qquad (3\text{-}58)$$

where F_Y is the tire lateral force, CALPHA is the cornering stiffness of the tire, and ALF is the corresponding tire slip angle. This relation implies that the lateral tire force is directly proportional to the slip angle. Consequently, wheel orientation to the road does not affect tire lateral force or tractive effort. This approximation allowed the full development of the system equations with less complexity. The impact of this approximation is discussed in chapter five.

A tire model more suitable to combined maneuver analysis was implemented after solutions with the linear model were understood. The nonlinear model was obtained from a reference in Wong [3] and initially developed by Dugoff [4]. This model uses a saturation function related to both the slip angle and longitudinal slip. This function is used to calculate both longitudinal and lateral forces produced by the tire. The equations as listed in Wong are:

$$F_X = \frac{CIS\ IS\ FS}{(1 - IS)} \qquad (3\text{-}59)$$

$$F_Y = \frac{CALPHA\ \tan(ALF)\ FS}{(1 - IS)} \qquad (3\text{-}60)$$

$$FS = \begin{cases} S\ (2 - S) & \text{for} \quad S < 1 \\ 1 & \text{for} \quad S > 1 \end{cases} \qquad (3\text{-}61)$$

$$S = \frac{MO\ F_z\ [1 - ER\ SSV\ (IS^2 + (\tan(ALF))^2)^{1/2}]\ (1 - IS)}{2\ [CI^2\ IS^2 + CALPHA^2\ (\tan(ALF))^2]^{1/2}} \quad (3\text{-}62)$$

where:

F_x = the longitudinal force generated by the tire;

F_y = the lateral force generated by the tire;

CIS = the longitudinal tire stiffness;

CALPHA = the lateral tire stiffness;

IS = the tire longitudinal slip;

ALF = the tire slip angle;

FS AND S = the saturation function;

MO = the coefficient of friction;

F_z = The normal load on the tire;

ER = reduction factor due to speed; and

SSV = steady state velocity.

These equations were developed to predict combined conditions of braking and cornering effort. Modifications to predict tractive and cornering effort involved only minor changes to the equations. The equations then become:

$$F_x = \frac{CIS\ IS\ FS}{(1 + IS)} \quad (3\text{-}63)$$

$$F_y = \frac{CALPHA\ \tan(ALF)\ FS}{(1 + IS)} \quad (3\text{-}64)$$

$$FS = \begin{cases} S\ (2 - S) & \text{for} \quad S < 1 \\ 1 & \text{for} \quad S > 1 \end{cases} \quad (3\text{-}65)$$

$$S = \frac{MO\ F_z\ [1 - ER\ SSV\ (IS^2 + (\tan(ALF))^2)^{1/2}]\ (1 + IS)}{2\ [CI^2\ IS^2 + CALPHA^2\ (\tan(ALF))^2]^{1/2}} \quad (3\text{-}66)$$

Some characteristic plots of force generation as predicted by the model are shown in Figures 3-10 and 3-11. Notice that this tire model is not comprehensive in that camber effects are not considered. This fact has direct implications on the amount of information the suspension model must generate.

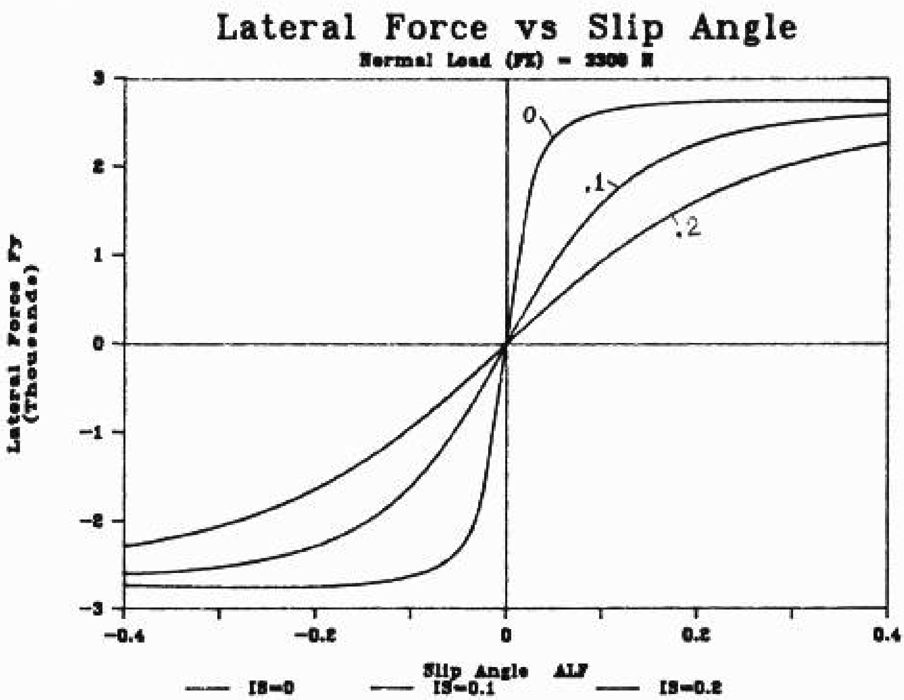


Figure 3-10 Lateral force characteristic of nonlinear tire model

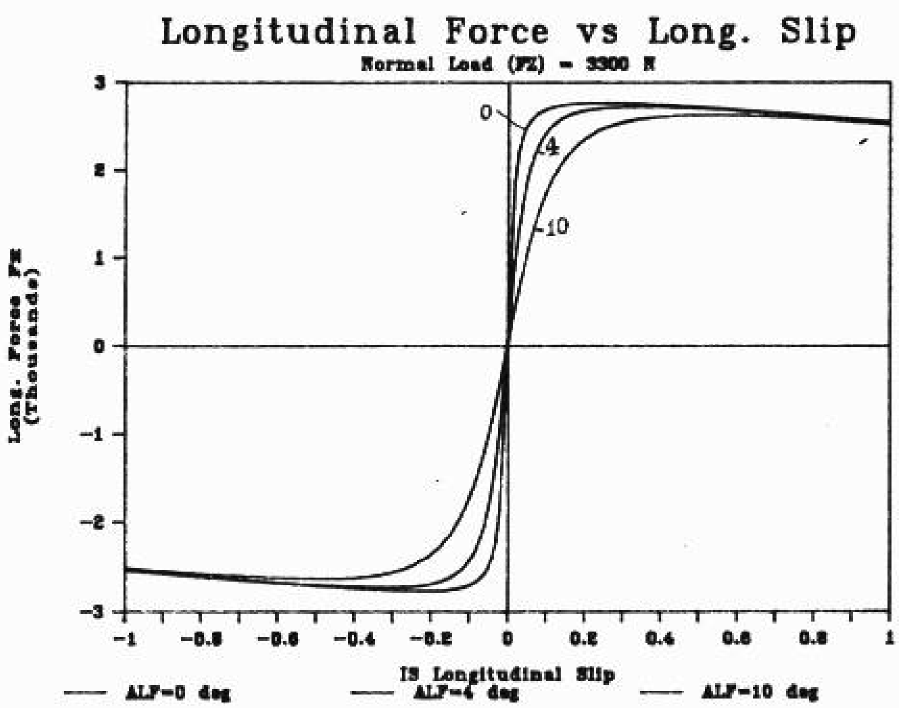


Figure 3-11 Long. force plot for nonlinear tire model

3.9 Suspension Model

The subtle nature of the implementation technique used for the suspension model makes the technique easy to overlook. This is unfortunate because the way the suspension model is implemented is the backbone of the general applicability of this vehicle model. A simple suspension model can be used if the tire model requires just the normal load and the slip angle. If the suspension model requires more information, or if anti-features are to be implemented then the suspension model must be more comprehensive. Therefore, this vehicle model's greatest flexibility is that the suspension model can be in virtually any form and only restricted by the requirements of the tire model.

The difficulty in modelling a suspension for a general purpose vehicle model is that the suspension geometry dictates how forces are distributed to the sprung mass. The equilibrium position of the vehicle under external loading depends on this distribution. Force distribution is controlled by the anti-features of the suspension as explained in Chapter Two. So, vehicle orientation to the road depends more on suspension geometry than external forces. By separating the calculation of wheel spindle displacements (suspension motion) into a subroutine, the nature of vehicle motion can be changed independent of the main vehicle equations. This allows simple interaction between the suspension (which controls sprung mass position) and sprung mass dynamics (which supplies force input to the suspension).

The suspension can be reduced to a simple spring of the form $F = K\,x$ at each wheel. However, this form does not represent the global restoring torques produced by anti-features. The other extreme is to include all suspension geometry in the development of the vehicle equations; but then the model is limited to one suspension type. The method used to overcome this problem is to use spindle forces as system unknowns and let spindle displacements be calculated off-line by the suspension routine. Then equations of motion can be derived using general vehicle dimensions. So, in the system equations the following parameters are used, negating the need for specific suspension geometry information:

K_F and K_B = spindle rates;

K_{PF} and K_{PB} = anti-roll bar rates; and

$Z_{S\_\_}$ =spindle displacements.

During initial computer model development the simplest linear tire and suspension models were used. The suspension equations are (3-34) through (3-37) and (3-38b).

3.10 Model Derivation Summary

The vehicle model consists of fourteen equations and fourteen unknowns with some auxiliary equations. The fourteen equations are derived in a piecewise manner and are difficult to distinguish from the pieces. To further outline exactly where these equations come from, they are listed in Table 1 under general headings along with the equation numbers involved. The fourteen unknowns are listed below for reference:

F_{XTFL}	F_{YTFL}	F_{ZFL}	BETA
F_{XTFR}	F_{YTFR}	F_{ZFR}	DELTA
F_{XBL}	F_{YBL}	F_{ZBL}	
F_{XBR}	F_{YBR}	F_{ZBR}	

In Chapter Four the equations derived in this chapter are manipulated in a minor way to improve the efficiency of the solution.

Table 1

General Summary of Equations Used for a Combined Maneuver Model

Equation('s) Description	Chapter Three Equation Numbers Involved
Six of the Fourteen are F = M a for the six dof's of the sprung mass	(3-3) thru (3-8) and (3-10) thru (3-24)
Three of the fourteen are drivetrain constraints for longitudinal motion	(3-43) thru (3-45)
One of the fourteen is the equation that constrains the sprung mass to a rigid body.	(3-42)
Four of the fourteen are the tire model used for each tire.	(3-58) or (3-59) to (3-66) (or any applicable model)
The rest are auxiliary equations.	(3-9), (3-26) thru (3-29) (3-34) thru (3-36) (3-39) thru (3-41) (3-46) thru (3-57)